EXTERIOR PRODUCTS OF HILBERT SPACES AND THE FREDOLM DETERMINANT

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Let H be a (separable) Hilbert space.[1] In this note we talk about the exterior products $\Lambda^k H$. The main application of this will be to define the Fredolm determinant $\det(1+A)$, for A trace class and to examine its properties.

1. Exterior Products

Consider the space $\Lambda^k H$ for $k \in N$, the (algebraic) vector space span of k-blades $\{v_1 \wedge \cdots \wedge v_k \colon v_1, \ldots, v_k \in H\}$. Formally, $\Lambda^k H$ is the quotient of the algebraic tensor product $H^{\otimes k}$ by the ideal generated by $\{v_1 \otimes \cdots \otimes v_k \colon v_i = v_j \text{ for some } i \neq j\}$. Observe that $\Lambda^k H$ is by definition characterized by the property that whenever $\Phi : H^k \to X$ is an alternating multilinear map of vector spaces, then there exists a unique map $\Lambda^k H \to X$ given by

$$\widetilde{\Phi}(v_1 \wedge \cdots \wedge v_k) = \Phi(v_1, \ldots, v_k)$$

on k-blades. We equip $\Lambda^k H$ with an inner product defined by

$$\langle v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k \rangle = \det(\langle v_i, w_j \rangle) \tag{1.1}$$

on k-blades and extending by linearity. Here, $\det(a_{ij})$ denotes the determinant of the matrix whose $(i,j)^{th}$ entry is a_{ij}. The space $\Lambda^k H$ is not a Hilbert space, so we hereafter replace $\Lambda^k H$ with its complition under this inner product, which is a Hilbert space. When we need to refer to the original, algebraic space, we will use the notation $\widetilde{\Lambda^k} H$.

The inner product (1.1) is not obviously well-defined, as k-blades don't have unique representations in $\widetilde{\Lambda^k} H$ (in fact a k-blade may be written as the sum of other k-blades!). We need to prove that it is well-defined.

Lemma 1.1. *The inner product* (1.1) *is well-defined on* $\widetilde{\Lambda^k} H$*, and hence defines an actual inner product.*

Proof. Fix $w_1, \ldots, w_k \in H$, and consider the map $\Phi : H^k \to \mathbf{C}$ defined by

$$\Phi(v_1, \ldots, v_k) = \det(\langle v_i, w_j \rangle).$$

Then Φ is is multilinear and alternating, and so by definition descends to a well-defined map from $\widetilde{\Lambda^k} H \to \mathbf{C}$. This shows that $\langle v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k \rangle$ is well-defined in the first argument. Since it is clearly conjugate symmetric, it is well-defned in the second argument, and so is well-defined overall. ☐

Lemma 1.2 (Properties of $\Lambda^k H$.)**.** *The following hold:*

[1]The separability assumption is mostly for notational convenience and to avoid having to talk about strongly convergent nets of projections rather than more pedestrian convergent sequences.

(i) Suppose that for $1 \le i \le k$, v_i^n is a sequence of vectors in H converging to $v_i \in H$. Then

$$v_1^n \wedge \cdots \wedge v_k^n \to v_1 \wedge \cdots \wedge v_k;$$

(ii) The span of the k-blades $\{v_1 \wedge \cdots \wedge v_k\}$ is dense in $\Lambda^k H$;

(iii) If $e_1, e_2, \ldots$ is an orthonormal basis of H, then $\Lambda^k H$ has an orthonormal basis of the form $\beta = \{e_{i_1} \wedge \cdots \wedge e_{i_k} \colon i_1 < \cdots < i_k\}$.

Proof. Let us prove (i). We compute

$$\|v_1^n \wedge \cdots \wedge v_k^n - v_1 \wedge \cdots \wedge v_k\|^2 = \det(\langle v_i^n, v_j^n\rangle) + \det(\langle v_i, v_j\rangle) - \det(\langle v_i^n, v_j\rangle) - \det(\langle v_i, v_j^n\rangle).$$

Since the determinant of a matrix is continuous in its entries, as $n \to \infty$ this converges to

$$\det(\langle v_i, v_j\rangle) + \det(\langle v_i, v_j\rangle) - \det(\langle v_i, v_j\rangle) - \det(\langle v_i, v_j\rangle) = 0.$$

(ii) is true since by definition $\widetilde{\Lambda^k} H$ is the span of k-blades, and this space is dense in its completion, $\Lambda^k H$.

Now let us prove (iii). It is clear that β is an orthonormal set. We show that it is a basis. Suppose $v_1, \ldots, v_k \in H$ and for all i

$$v_i = \lim_{n\to\infty} w_i^n$$

for $w_i^n \in \operatorname{span}\{e_1, e_2, \ldots\}$ is in the vector space span (i.e. is a finite linear combination). From (i), we know that

$$w_1^n \wedge \cdots \wedge w_k^n \to v_1 \wedge \cdots v_k.$$

Each k-blade $w_1^n \wedge \cdots \wedge w_k^n$ belongs to $\operatorname{span}\beta$, so this shows that $\operatorname{span}\beta$ is dense in $\Lambda^k H$, which is sufficient. □

We now show that a bounded linear map A can be used to define an operator $\Lambda^k A$ on $\Lambda^k H$, and that this assignment is functorial.

Theorem 1.3. *Let $A : H \to H$ be bounded. Then for each k there exists a unique bounded operator $\Lambda^k A : \Lambda^k H \to \Lambda^k H$ such that $\Lambda^k A$ acts on k-blades by*

$$(\Lambda^k A)(v_1 \wedge \cdots v_k) = Av_1 \wedge \cdots \wedge Av_k.$$

Furthermore, $\|\Lambda^k A\| \le \|A\|^k$, and the map $\Lambda^k : B(H) \to B(\Lambda^k H)$ is continuous. Explicitly, for $A, B \in B(H)$ (and $k \ge 1$)

$$\|\Lambda^k A - \Lambda^k B\| \le k\|A - B\| \max(\|A\|, \|B\|)^{k-1}. \tag{1.2}$$

The map Λ^k is functorial in the following sense:

(i) $\Lambda^k(AB) = \Lambda^k A \Lambda^k B$ for A, B bounded;

(ii) if A is invertible, then $\Lambda^k A$ is invertible with inverse $\Lambda^k A^{-1}$;

(iii) $(\Lambda^k A)^* = \Lambda^k A^*$;

(iv) if $\Pi : H \to K$ is the orthogonal projection onto K, then $\Lambda^k \Pi$ is the orthogonal projection onto $\Lambda^k K$, the closure of the span of k-blades $\{v_1 \wedge \cdots \wedge v_k \colon v_i \in K, 1 \le i \le k\}$;

(v) if A is positive, then $\Lambda^k A$ is positive;

(vi) $|\Lambda^k A| = \Lambda^k |A|$.

If A is additionally trace class, then $\Lambda^k A$ is also trace class, and

$$\|\Lambda^k A\|_1 \leq \frac{\|A\|_1^k}{k!}.$$

Futhermore, the map $\Lambda^k : \ell^1(H) \to \ell^1(\Lambda^k H)$ is continuous, with explicit bounds for A, B trace class (for $k \geq 1$)

$$\|\Lambda^k A - \Lambda^k B\|_1 \leq \|A - B\|_1 \frac{\max(\|A\|_1, \|B\|_1)^{k-1}}{(k-1)!}. \tag{1.3}$$

To prove this, we need the following technical lemma:

Lemma 1.4. *Let H be a Hilbert space and suppose $X \subseteq H$ is dense. Let A_n be a sequence of uniformly bounded operators such that $A_n x$ converges pointwise for each $x \in X$. Then A_n converges strongly to a bounded operator A, and $\|A\| \leq \limsup \|A_n\|$.*

Proof. We show that for all $v \in H$, $A_n v$ is Cauchy, and thus A_n converges strongly to a linear map A. Fix $\varepsilon > 0$. By density and uniform boundedness, there exists $x \in X$ such that for all $n \in \mathbf{N}$, $\|A_n v - A_n x\| < \varepsilon/3$. Now for N large, if $n, m > N$, we may assume that $\|A_n x - A_m x\| < \varepsilon/3$. Thus if $n, m > N$

$$\|A_n v - A_m v\| \leq \|A_n v - A_n x\| + \|A_n x - A_m x\| + \|A_m v - A_m x\| < \varepsilon.$$

For $v \in H$, and $\varepsilon > 0$, again choose x with $\|A_n x - A_n v)\| \leq \varepsilon$. Then

$$\|Av\| = \lim_{n \to infty} \|A_n v\| \leq \limsup_{n \to \infty} \|A_n v - A_n x\| + \|A_n x\| \leq \varepsilon + \limsup_{n \to \infty} \|A\| \|x\|.$$

Since $\|x\| \leq \varepsilon + \|v\|$, it follows that

$$\|Av\| \leq (1 + \limsup_{n \to \infty} \|A\|)(\varepsilon) + \limsup_{n \to \infty} \|A\| \|v\|.$$

Taking $\varepsilon \to 0$ shows that $\|Av\| \leq \limsup_{n \to \infty} \|A\| \|v\|$, which shows that A is bounded and $\|A\| \leq \limsup \|A_n\|$. □

Proof of theorem 1.3. This theorem has many different parts, so we prove them separately. **Part 1: uniqueness and functoriality.** Since the span of k-blades is dense, uniqueness follows immediately. By density and linearity, functoriality will follow if we can check each statement on a basis. For (i), observe that for any k-blade $v_1 \wedge \cdots \wedge v_k$,

$$\begin{aligned}\Lambda^k(AB)(v_1 \wedge \cdots \wedge v_k) &= (ABv_1) \wedge \cdots \wedge (ABv_k) \\ &= \Lambda^k A((Bv_1) \wedge \cdots \wedge (Bv_k)) \\ &= \Lambda^k A \Lambda^k B(v_1 \wedge \cdots \wedge v_k).\end{aligned}$$

Property (ii) follows from (i), since

$$\Lambda^k A^{-1} \Lambda^k A = \Lambda^k 1 = \Lambda^k A \Lambda^k A^{-1},$$

and $\Lambda^k 1$ is certainly the identity since it maps any k-blade to itself. For (iii), observe that for any other k-blade $w_1 \wedge \cdots \wedge w_k$,

$$\begin{aligned}\langle \Lambda^k A(v_1 \wedge \cdots \wedge v_k), w_1 \wedge \cdots \wedge w_k \rangle &= \det(\langle Av_i, w_j \rangle) = \det(\langle v_i, Aw_j \rangle) \\ &= \langle v_1 \wedge \cdots \wedge v_k, \Lambda^k A w_1 \wedge \cdots \wedge w_k.\end{aligned}$$

For (iv), observe that $\Lambda^k\Pi$ is self-adjoint (from (iii)) and idempotent (from (i)). Thus $\Lambda^k\Pi$ is the orthogonal projection onto its range. Certainly $\Lambda^k K \subseteq \text{range}(\Lambda^k\Pi)$, since $\Lambda^k\Pi$ acts as the identity on the wedge product of vectors in K. We now show that its range is contained in $\Lambda^k K$. If $v = v_1 \wedge \cdots \wedge v_k$ is a k-blade, then we may write $v_i = u_i + w_i$ where $u_i \in K$ and $w_i \perp K$. Thus

$$v = u_1 \wedge \cdots \wedge u_k + w,$$

where w is a sum of wedges at least one of whose factors is orthogonal to K. Thus

$$\Lambda^k\Pi v = u_1 \wedge \cdots \wedge u_k + 0 \in \Lambda^k K.$$

It follows that the range of $\Lambda^k\Pi$ on the span of k-blades is contained in $\Lambda^k K$, and hence the range of $\Lambda^k\Pi$ on all of $\Lambda^k H$ is contained in $\Lambda^k H$, since the span of k-blades is dense and $\Lambda^k K$ is closed by definition.

For (v), first assume that A is compact. Suppose $e_1, e_2, \ldots$ is an orthonormal basis for H of eigenvectors of $|A|$. Then $\{e_{i_1} \wedge \cdots \wedge e_{i_k} : i_1 < \cdots < i_k\}$ is an orthonormal basis of eigenvectors of $\Lambda^k A$. Since each associated eigenvalue is positive, it follows that $\Lambda^k A$ is positive. If A is not compact, then fix any orthonormal basis $e_1, e_2, \ldots$ of H, and let Π_n be the orthogonal projection onto $\text{span}\{e_1, \ldots, e_n\}$. Then $\Pi_n A \Pi_n$ is positive, and so $\Lambda^k\Pi_n \Lambda^k A \Lambda^k \Pi_n$ is positive. The operator $\Lambda_k \Pi_n$ is by (iv) the orthogonal projection onto $\text{span}\{e_{i_1} \wedge e_{i_k} : i_1 < \cdots i_k \le n\}$, and thus converges strongly to 1. Thus $\Lambda^k\Pi_n \Lambda^k A \Lambda^k \Pi_n$ converges strongly to $\Lambda^k A$. Since a strong limit of positive operators is positive, $\Lambda^k A$ is also positive.

For (vi), observe first that

$$(\Lambda^k |A|)^2 = \Lambda^k |A|^2 = \Lambda^k A^* A = (\Lambda^k A)^* \Lambda^k A,$$

and $\Lambda^k |A|$ is positive. Thus $\Lambda^k |A|$ is a positive square root of $(\Lambda^k A)^* \Lambda^k A = |\Lambda^k A|^2$, and thus must coincide with $|\Lambda^k A|$.[2]

Part 2: Existence. Now let us show existence. We first suppose that A is positive and compact. Let $e_1, e_2, \ldots$ be an orthonormal basis of eigenvctors of A, and suppose $Ae_i = \lambda_i e_i$. Suppose λ_1 is the largest eigenvalue. Let $\beta = \{e_{i_1} \wedge \cdots \wedge e_{i_k} : i_1 < \cdots < i_k\}$. We first define a map $B : \text{span}\,\beta \to \Lambda^k H$, and then show it is bounded, and thus B extends to a bounded map $B : \Lambda^k H \to \Lambda^k H$. We then show that

$$B(v_1 \wedge \cdots \wedge v_k) = Av_1 \wedge \cdots \wedge Av_k,$$

and thus we can define $\Lambda^k A = B$. For $\alpha = (\alpha_1, \ldots, \alpha_k)$ an increasing k-tuple, set $e_\alpha = e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_k}$, and $\lambda_\alpha = \lambda_{\alpha_1} \cdots \lambda_{\alpha_k}$. Define $B(e_\alpha) = \lambda_\alpha e_\alpha$, and then extend by linearity. Thus, if $v = \sum a_\alpha e_\alpha$ is a finite linear combination,

$$\|Bv\|^2 = \sum |a_\alpha|^2 \|Be_\alpha\|^2 = \sum |a_\alpha|^2 \lambda_\alpha^2 \le \lambda_1^{2k} \|v\|^2,$$

and so B extends to a bounded operator. In fact, this shows that $\|B\| \le \lambda_1^k = \|A\|^k$.

If $v_1, \ldots, v_k \in H$ are finite linear combinations of the e_i, then it is easy to check that

$$B(v_1 \wedge \cdots \wedge v_k) = Av_1 \wedge \cdots \cdots \wedge Av_k.$$

[2] Indeed, if P and Q are positive operators on a Hilbert space, and $Q^2 = P$, then $Q = \sqrt{P}$. To show this, suppose $a > 0$ is large enough so that $\sigma(P) \subseteq [0, a]$ and $\sigma(Q) \subseteq [0, \sqrt{a}]$. Suppose $p_n(x)$ are polynomials converging to $\sqrt{x}$ uniformly on $[0, a]$. Then $p_n(Q^2) = p_n(P) \to \sqrt{P}$. On the other hand, $p_n(x^2) \to x$ on $[0, \sqrt{a}]$, and so $p_n(Q^2) \to Q$.

Indeed, suppose $v_i = \sum a_i^j e_j$ for all i. Let N be the large index such that a_i^N is nonzero for some i. Let T_k denote the set of injective maps from $\{1, \ldots, k\} \to \{1, \ldots, N\}$. Then

$$\begin{aligned} B(v_1 \wedge \cdots \wedge v_k) &= B\left(\sum_{\sigma \in T_k} \prod_{i=1}^{k} a_i^{\sigma(i)} \bigwedge_{i=1}^{k} e_{\sigma(i)}\right) \\ &= \sum_{\sigma \in T_k} \prod_{i=1}^{k} a_i^{\sigma(i)} \bigwedge_{i=1}^{k} \lambda_i e_{\sigma(i)} \\ &= \sum_{\sigma \in T_k} \prod_{i=1}^{k} a_i^{\sigma(i)} \bigwedge_{i=1}^{k} A e_{\alpha(\sigma)_i} \\ &= Av_1 \wedge \cdots \cdots \wedge Av_k. \end{aligned}$$

Now if $v_1, \ldots, v_k$ are not finite linear combinations, then we can write them as a limit of finite linear combinations, and use the fact that B and A are bounded, together with lemma 1.2.

Now assume that A is positive, but not compact. If $e_1, e_2, \ldots$ is any orthonormal basis of H, let Π_n denote the orthogonal projection onto $\operatorname{span}\{e_1, \ldots, e_n\}$. Then $\Pi_n A \Pi_n$ is positive and compact, and so $\Lambda^k(\Pi_n A \Pi_n)$ exists. Using lemma 1.2 and the definition of $\Lambda^k(\Pi_n A \Pi_n)$ on k-blades, it follows that for any k-blade $v_1 \wedge \cdots \wedge v_k$

$$\Lambda^k(\Pi_n A \Pi_n) v_1 \wedge \cdots \wedge v_k = (\Pi_n A \Pi_n v_1) \wedge \cdots \wedge (\Pi_n A \Pi_n A v_k) \to Av_1 \wedge \cdots \wedge Av_k.$$

Thus by linearity $\Lambda^k(\Pi_n A \Pi_n)$ converges pointwise on the span of k-blades. Also,

$$\|\Lambda^k \Pi_n A \Pi_n\| \leq \|\Pi_n A \Pi_n\|^k \leq \|A\|^k$$

for each n. Thus, since the span of k-blades is dense, by lemma 1.4, $\Lambda^k(\Pi_n A \Pi_n)$ converges strongly to some operator B. Since we have already shown that

$$B(v_1 \wedge \cdots \wedge v_k) = \lim_{n \to \infty} \Lambda^k(\Pi_n A \Pi_n)(v_1 \wedge \cdots \wedge v_k) = Av_1 \wedge \cdots \cdots \wedge Av_k$$

for any k-blade, we may set $\Lambda^k A = B$.

Now let A be a partial isometry. Let $e_1, e_2, \ldots$ be an orthonormal basis of H which is the result of taking the union of an orthonormal basis of $\ker A$ and $\ker A^\perp$ (and then relabelling), and let β be as above. As above, we first define a map $B : \operatorname{span} \beta \to \Lambda^k H$, show it is bounded, and that it behaves correctly on k-blades. Define

$$B(e_\alpha) = Ae_{\alpha_1} \wedge \cdots \wedge Ae_{\alpha_n}.$$

If α and α' are increasing k-tuples, then

$$\langle Be_\alpha, Be_{\alpha'} \rangle = \det(\langle A^* A e_{\alpha_i}, e_{\alpha'_j} \rangle).$$

Now $A^* A$ is precisely the projection onto $\ker A^\perp$. Thus, if any $e_{\alpha_i} \in \ker A$, $\langle Be_\alpha, Be'_\alpha \rangle = 0$. Otherwise (i.e. all e_{α_i} are in $\ker A^\perp$), it is equal to

$$\det(\langle e_{\alpha_i}, e_{\alpha'_j} \rangle) = \langle e_\alpha, e_{\alpha'} \rangle.$$

Now, if $v = \sum a_\alpha e_\alpha$ is a finite linear combination, let S be the collection of those α such all $e_{\alpha_i} \in \ker A^\perp$. Then

$$\|Bv\|^2 = \sum_{\alpha, \alpha'} a_\alpha \overline{a_{\alpha'}} \langle Be_\alpha, Be'_\alpha \rangle$$

$$= \sum_{\alpha \in S, \alpha'} a_\alpha \overline{a_{\alpha'}} \langle e_\alpha, e_{\alpha'} \rangle$$
$$= \sum_{\alpha \in S} |a_\alpha|^2 \leq \|v\|^2.$$

It follows that B is bounded and has norm precisely 1 (which is of course also $\|A\|^k$). The proof that B behaves correctly on k-blades is the same as in the case that A is compact and positive. Thus in this case, too, can we set $\Lambda^k A = B$.

Now for the general case. Suppose A is bounded. Write $A = U|A|$ the polar decomposition, where U is a partial isometry and $|A|$ is positive. Define

$$\Lambda^k A = \Lambda^k U \Lambda^k |A|,$$

both factors of which exist. We need to show that $\Lambda^k A$ behaves properly on k-blades. But this is obvious, as for any k-blade $v_1 \wedge \cdots \wedge v_k$,

$$\Lambda^k U \Lambda^k |A| v_1 \wedge \cdots \wedge v_k = \Lambda^k U(|A|v_1 \wedge \cdots \wedge |A|v_k) = (U|A|v_1) \wedge \cdots \wedge (U|A|v_k).$$

Certainly

$$\|\Lambda^k A\| \leq \|\Lambda^k U\| \|\Lambda^k |A|\| \leq \||A|\|^k = \|A\|^k.$$

Part 4: Continuity. Suppose A, B are bounded operators. Let $e_1, e_2, \ldots$ be an orthonormal basis of H, and let Π_n denote the projection on $\{e_1, \ldots, e_n\}$. Since $\Lambda^k \Pi_n$ is the projection onto $\operatorname{span}\{e_{i_1} \wedge e_{i_k} \colon i_1 < \cdots i_k \leq n\}$, $\Lambda^k \Pi_n (\Lambda^k A - \Lambda^k B) \Lambda^k \Pi_n$ converges in the strong operator topology $\Lambda^k A - \Lambda^k B$. Since the operator norm is lower semicontinuous in the strong operator topology, it suffices to prove (1.2) with A and B replaced by $\Pi_n A \Pi_n$ and $\Pi_n B \Pi_n$, respectively. In other words, we may assume that H is finite-dimensional with $\dim H = n$ (and hence $k \leq n$ since the spaces $\Lambda^k H = 0$ for $k > n$). For $t \in [0, 1]$, let $C(t) = tA + (1-t)B$. For α an increasing k-tuple and $v \in \Lambda^k H$, define

$$\gamma_{\alpha,v}(t) = \langle \Lambda^k C(t) e_\alpha, v \rangle,$$

which is smooth on $[0, 1]$ (since v may be expanded in a finite basis of $\Lambda^k H$). In particular,

$$\langle (\Lambda^k A - \Lambda^k B) e_\alpha, v \rangle = \gamma_{\alpha,v}(1) - \gamma_{\alpha,v}(0) = \int_0^1 \gamma'_{\alpha,v}(t)\, dt. \tag{1.4}$$

We now compute $\gamma'_{\alpha,v}$. For $1 \leq i \leq k$, denote by $\widehat{e_{\alpha_i}}$ the wedge of all e_{α_i} (in order) *except* e_{α_i}. Write $A - B = V|A - B|$. Since H is finite-dimensional, we may assume that V is unitary. We may also assume (by the spectral theorem) that the basis $\{e_1, \ldots, e_n\}$ of H is a basis of eigenvectors for $|A - B|$, with eigenvalues $\lambda_i \geq 0$. Write $Ve_i = f_i$, so that $\{f_1, \ldots, f_n\}$ is an orthonormal basis. Use the notation $f_\alpha = f_{\alpha_1} \wedge \cdots \wedge f_{\alpha_\ell}$ for an increasing ℓ-tuple α. Using that the wedge product is continuous, it is easy to check that

$$\gamma'_{\alpha,v}(t) = \sum_{i=1}^k \langle e_{\alpha_1} \wedge \cdots \wedge_{e_{\alpha_{i-1}}} \wedge C'(t) e_{\alpha_i} \wedge e_{\alpha_{i+1}} \wedge \cdots \wedge e_{\alpha_k}, v \rangle$$
$$= \sum_{i=1}^k (-1)^{i+1} \langle (A - B) e_{\alpha_i} \wedge \Lambda^{k-1} C(t) \widehat{e_{\alpha_i}}, v \rangle$$

$$= \sum_{i=1}^{k} (-1)^{i+1} \lambda_{\alpha_i} \langle f_{\alpha_i} \wedge \Lambda^{k-1} C(t) \widehat{e_{\alpha_i}}, v \rangle.$$

For all j, the wedge map $f_j \wedge \colon \Lambda^{k-1} H \to \Lambda^k H$ has norm 1. Let ι_{f_j} denote its adjoint which also has norm 1. Then we can rewrite the previous display as

$$\gamma'_{\alpha,v}(t) = \sum_{i=1}^{k} (-1)^{i+1} \lambda_{\alpha_i} \langle \Lambda^{k-1} C(t) \widehat{e_{\alpha_i}}, \iota_{f_{\alpha_i}} v \rangle. \tag{1.5}$$

For $v \in \Lambda^k H$, let v_α be the coefficients in the expansion $v = \sum_\alpha v_\alpha e_\alpha$. Then, from (1.4),

$$\begin{aligned} \|\Lambda^k A - \Lambda^k B\| &= \sup_{\|v\|=\|w\|=1} \left| \sum_\alpha v_\alpha \langle (\Lambda^k A - \Lambda^k B) e_\alpha, w \rangle \right| \\ &\leq \sup_{\|v\|=\|w\|=1} \int_0^1 \left| \sum_\alpha v_\alpha \gamma'_{\alpha,w}(t)\ dt \right| dt. \end{aligned} \tag{1.6}$$

Fix some v, w with $\|v\| = \|w\| = 1$. Plugging in (1.5) for $\gamma'_{\alpha,w}(t)$ and applying Cauchy-Schwarz inequality yields

(1.7)

$$\left| \sum v_\alpha \gamma'_{\alpha,w}(t)\ dt \right| = \left| \sum_\alpha v_\alpha \gamma'_{\alpha,v}(t) \right| \leq \left(\sum_{\alpha,i} |v_\alpha|^2 |\lambda_{\alpha_i}|^2 \right)^{1/2} \left(\sum_{\alpha,i} |\Lambda^{k-1} C(t) \widehat{e_{\alpha_i}}, \iota_{f_{\alpha_i}} w \rangle|^2 \right)^{1/2}.$$

The first factor is bounded by

$$\sqrt{k} (\sup \lambda_i) \|v\| = \sqrt{k} \|A - B\|.$$

For the second, we may rewrite the sum instead over all pairs (j, β), where $1 \leq j \leq n$, and β is an increasing $(k-1)$-tuple none of whose terms is j. This yields

$$\sum_{\alpha,i} |\langle \Lambda^{k-1} C(t) \widehat{e_{\alpha_i}}, \iota_{f_{\alpha_i}} v \rangle|^2 = \sum_{\beta,j} |\langle e_\beta, \Lambda^{k-1} C(t)^* \iota_{f_j} w \rangle|^2.$$

Taking the sum first over β, one bounds this by

$$\sum_j \|\Lambda^{k-1} C(t)^* \iota_{f_j} w\|^2 \leq \|\Lambda^{k-1} C(t)^*\|^2 \sum_j \|\iota_{f_j} w\|^2.$$

The first factor is bounded by $\|C(t)^*\|^{2(k-1)} = \|C(t)\|^{2(k-1)}$. For the second factor, expand $w = \sum w_\alpha f_\alpha$. Notice that $\iota_{f_j} f_\alpha = 0$ if j is not a term in α. Otherwise, $\iota_{f_j} f_\alpha = \pm f_{\alpha'}$, where α' is the increasing $(k-1)$-tuple obtained from α by removing j (the sign depends on j and α). Thus, $\langle \iota_{f_j} f_\alpha, \iota_{f_j} f_\beta \rangle = \delta_{\alpha=\beta}$, the Kronecker δ, and

$$\|\iota_{f_j} w\|^2 = \sum_{\alpha,\beta} w_\alpha \overline{w_\beta} \langle \iota_{f_j} e_\alpha, \iota_{f_j} e_\beta \rangle = \sum_{\alpha \ni j} |w_\alpha|^2,$$

where the sum ranges over all those α one of whose terms is j. Thus

$$\sum_j \|\iota_{f_j} w\|^2 = \sum_j \sum_{\alpha \ni j} |w_\alpha|^2.$$

In this sum, each term $|w_\alpha|^2$, for an increasing k-tuple α, appears precisely k times: once for each $j = \alpha_i$, $1 \le i \le k$. We conclude that

$$\sum_j \|\iota_{f_j} w\|^2 = k \sum_\alpha |w_\alpha|^2 = k\|w\|^2 = k.$$

Putting it all together, the second factor on the last line of (1.7) is bounded by $\|C(t)\|^{j-1}\sqrt{k}$, and recalling the bound on the first factor, (1.7) is bounded by

$$k\|A - B\|\|C(t)\|^k.$$

Now

$$\|C(t)\| \le (1-t)\|A\| + t\|B\| \le \max(\|A\|, \|B\|).$$

Hence, from (1.6),

$$\|\Lambda^k A - \Lambda^k B\| \le \int_0^1 k\|A - B\| \max(\|A\|, \|B\|)^k \, dt \le k\|A - B\| \max(\|A\|, \|B\|)^{k-1},$$

which is the desired bound.

Part 5: Trace class operators. Now let us suppose A is trace class. We prove that $\Lambda^k A$ is trace class, i.e. $|\Lambda^k A|$ is trace class. We know that $|\Lambda^k A| = \Lambda^k |A|$, so replacing A with $|A|$, we can assume that A is positive. Since $|A|$ is compact, by the spectral theorem we can find $e_1, e_2, \ldots$, an orthonormal basis of eigenvectors of A, and suppose $Ae_i = \lambda_i e_i$. Let $\beta = \{e_{i_1} \wedge \cdots \wedge e_{i_k} : i_1 < \cdots < i_k\}$. Then

$$\begin{aligned}
\mathrm{Tr}(\Lambda^k A) &= \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k} \\
&= \lim_{n\to\infty} \sum_{i_1 < \cdots < i_k \le n} \lambda_{i_1} \cdots \lambda_{i_k} \\
&= \lim_{n\to\infty} \sum_{i_1 < \cdots < i_k \le n} \frac{1}{k!} \sum_{\sigma \in S_k} \lambda_{i_{\sigma(1)}} \cdots \lambda_{i_{\sigma(k)}} \\
&= \lim_{n\to\infty} \frac{1}{k!} \sum_{(i_1,\ldots,i_k), i_k \le n, \text{ has distinct entries}} \lambda_{i_1} \cdots \lambda_{i_k} \\
&\le \lim_{n\to\infty} \frac{1}{k!} \sum_{(i_1,\ldots,i_k), i_k \le n, \text{ a } k\text{-tuple}} \lambda_{i_1} \cdots \lambda_{i_k} \\
&= \lim_{n\to\infty} \frac{1}{k!} \left(\sum_{i=1^n} \lambda_i \right)^k \\
&= \frac{\mathrm{Tr}(A)^k}{k!}.
\end{aligned}$$

Thus A is trace class and $\|A\|_1 \le \frac{\|A\|_1^k}{k!}$.

Part 6: Continuity in the trace norm. The proof starts very similarly to part 4, the proof of the continuity in the operator norm, and we use the same notation. Suppose A, B are trace-class operators. Let $e_1, e_2, \ldots$ be an orthonormal basis of H, and let Π_n denote the projection on $\{e_1, \ldots, e_n\}$. Recall that $\Lambda^k \Pi_n$ is the projection onto $\mathrm{span}\{e_{i_1} \wedge e_{i_k} : i_1 < \cdots i_k \le n\}$. We will show below in lemma 1.5 that this means that, $\Pi_n A \Pi_n \to A$,

$\Pi_n B\Pi_n \to B$, $\Lambda^k\Pi_n A\Pi_n \to \Lambda^k A$, $\Lambda^k\Pi_n B\Pi_n \to \Lambda^k B$, all in the trace norm. Thus, it suffices to prove (1.3) with A and B replaced by $\Pi_n A\Pi_n$ and $\Pi_n B\Pi_n$, respectively. In other words, we may assume that H is finite-dimensional with $\dim H = n$.

Let $C(t)$, $\gamma_{\alpha,v}(t)$, f_α, λ_i be as in part 4. Write $\Lambda^k A - \Lambda^k B = U|\Lambda^k A - \Lambda^k B|$ for the polar decomposition, so that $|\Lambda^k A - \Lambda^k B| = U^*(\Lambda^k A - \Lambda^k B)$. Then, from (1.4),

$$\begin{aligned} \|\Lambda^k A - \Lambda^k B\|_1 &= |\operatorname{Tr}(U^*(\Lambda^k A - \Lambda^k B))| \\ &= \left|\sum_\alpha \langle \Lambda^k A - \Lambda^k B)e_\alpha, Ue_\alpha\rangle\right| \\ &\le \int_0^1 \left|\sum_\alpha \gamma'_{\alpha,Ue_\alpha}(t)\right| \, dt. \end{aligned} \tag{1.8}$$

We will bound the integrand uniformly. Plugging in (1.5) for the integrand yields

$$\sum_\alpha \gamma'_{\alpha,Ue_\alpha}(t) = \sum_{\alpha,i} (-1)^{i+1}\lambda_{\alpha_i}\langle \Lambda^{k-1}C(t)\widehat{e_{\alpha_i}}, \iota_{f_{\alpha_i}} Ue_\alpha\rangle. \tag{1.9}$$

We may rewrite the sum instead over all pairs (j,β), where $1 \le j \le n$, and β is an increasing $(k-1)$-tuple none of whose terms is j. To do so, notice that $e_\alpha = (-1)^{i+1} e_{\alpha_i} \wedge \widehat{e_{\alpha_i}}$. Thus, the sum is equal to

$$\sum_\alpha \gamma'_{\alpha,Ue_\alpha}(t) = \sum_{j,\beta} \lambda_j \langle \Lambda^{k-1}C(t)e_\beta, \iota_{f_j} U(e_j \wedge e_\beta)\rangle. \tag{1.10}$$

For j fixed, let $U_j : \Lambda^{k-1}H \to \Lambda^{k-1}H$ be the map $w \mapsto \iota_{f_j} U(e_j \wedge w)$, which has norm at most 1. Let $\Gamma_j \colon H \to H$ be the projection off of e_j. Then $\Lambda^{k-1}\Gamma_j e_\beta = e_\beta$ precisely when j is not an index in β, and is 0 otherwise. Then, for j fixed, the the sum over β is just

$$\operatorname{Tr}((\Lambda^{k-1}\Gamma_j U_j^* \Lambda^{k-1}C(t)\Lambda^{k-1}\Gamma_j),$$

which is bounded by

$$\|\Lambda^{k-1}\Gamma_j\|^2 \|\Lambda^{k-1}U_j^*\| \|\Lambda^{k-1}C\|_1 \le \frac{\|C\|_1^{k-1}}{(k-1)!},$$

using the bounds we have proven previously. Thus, (1.10) is bounded by

$$\begin{aligned} \left|\sum_\alpha \gamma'_{\alpha,Ue_\alpha}(t)\right| = \left|\sum_j \lambda_j \operatorname{Tr}((\Lambda^{k-1}\Gamma_j U_j^* \Lambda^{k-1}C(t)\Lambda^{k-1}\Gamma_j)\right| &\le \left(\sum_j \lambda_j\right) \frac{\|C(t)\|_1^{k-1}}{(k-1)!} \\ &= \|A-B\|_1 \frac{\|C(t)\|_1^{k-1}}{(k-1)!.} \end{aligned} \tag{1.11}$$

However,

$$\|C(t)\|_1 \le (1-t)\|A\|_1 + t\|B\|_1 \le \max(\|A\|_1, \|B\|_1).$$

Therefore, returning to (1.8) and using (1.11)

$$\|\Lambda^k A - \Lambda^k B\| \le \int_0^1 \left|\sum_\alpha \gamma'_{\alpha,Ue_\alpha}(t) \ dt\right| \ dt$$

$$\leq \int_0^1 \|A-B\|_1 \frac{\max(\|A\|_1, \|B\|_1)^{k-1}}{(k-1)!} dt$$
$$\leq \|A-B\|_1 \frac{\max(\|A\|_1, \|B\|_1)^{k-1}}{(k-1)!},$$

which is the desired bound. □

We now prove the lemma about convergence in the trace class, which we will also use later.

Lemma 1.5. *Let H be a Hilbert space, and let $E_1 \subseteq E_2 \subseteq \cdots$ be a family of strictly increasing finite-dimensional subspaces of H whose closure is dense. Let Π_i be the projection on E_i. Then $A(1-\Pi_n) \to 0$ and $(1-\Pi_n)A \to 0$ in the trace class norm. In particular, $\Pi_n A \Pi_n \to A$ in the trace-class norm.*

Proof. The second claim follows from the first by bounding

$$\|\Pi_n A \Pi_n - A\|_1 \leq \|(\Pi_n - 1)A\| \|\Pi_n\| + \|A(\Pi_n - 1)\|_1.$$

The statement for $(1-\Pi_n)A$ follows from that for $A(1-\Pi_n)$ by taking adjoints.

Write $A = U|A|$ and $A(1-\Pi_n) = V|A(1-\Pi_n)|$ for the polar decompositions. Then

$$|A(1-\Pi_n)| = V^*U|A|(1-\Pi_n) = (V^*U|A|^{1/2})(|A|^{1/2}(1-\Pi_n)).$$

Set $W = V^*U$. We may pick an orthonormal basis $\{e_1, \dots, e_{m_1}\}$ of E_1, extend it to an orthonormal basis $\{e_1, \dots, e_{m_2}\}$ of E_2, etc, obtaining an orthonormal basis $e_1, e_2, \dots$ of H, such that for $m_i = \dim E_i$, $\{e_1, \dots, e_{m_i}\}$ is an orthonormal basis for E_i. Then

$$\|A(1-\Pi_n)\|_1 = |\operatorname{Tr}(|A(1-\Pi_n)|)| = \left|\sum_{i=1}^{\infty} \langle |A|^{1/2}(1-\Pi_n)e_i, |A|^{1/2}W^*e_i\rangle\right|$$
$$= \left|\sum_{i=m_n}^{\infty} \langle |A|^{1/2}e_i, |A|^{1/2}W^*e_i\rangle\right|$$
$$\leq \left(\sum_{i=m_n}^{\infty} \||A|^{1/2}e_i\|^2\right)^{1/2} \left(\sum_{i=m_n}^{\infty} \||A|^{1/2}W^*e_i\|^2\right)^{1/2}.$$

The square of the second factor is bounded, uniformly in n, by

$$\sum_{i=1}^{\infty} \||A|^{1/2}W^*e_i\|^2 = \operatorname{Tr}(W|A|W^*) \leq \|A\|_1.$$

The square of the first factor is

$$\sum_{i=m_n}^{\infty} \langle\langle |A|e_i, e_i\rangle,$$

which goes to 0 as $n \to \infty$. □

2. The Fredholm determinant

We can now define the Fredholm determinant.

Definition 2.1. Suppose $A : H \to H$ is trace class. Then define

$$\det_{\mathrm{Frd}}(1 + A) = \sum_{k=0}^{\infty} \mathrm{Tr}(\Lambda^k A),$$

interpreting $\mathrm{Tr}(\Lambda^0 A) = 1$. This makes sense since by theorem 1.3 $|\mathrm{Tr}(\Lambda^k A)| \leq \frac{\|A\|_1^k}{k!}$ for all k, and hence the defining series is absolutely summable.

Let us check that this agrees with the usual definition in the case that H is finite-dimensional. In fact,

Proposition 2.2. *Suppose $K \subseteq H$ is finite-dimensional, and $A = \Pi A \Pi$, where Π is the orthogonal projection onto K. Then, with $\det_{\mathrm{us}}$ interpreted as the usual determinant of a linear map between fininite dimensional spaces,*

$$\det_{\mathrm{us}}((1 + A)|_K) = \det_{\mathrm{Frd}}(1 + A).$$

Proof. Suppose $\dim K = n$. Fix $k > n$, and a k-blade $v = v_1 \wedge \cdots \wedge v_k$. Write $v_i = u_i + w_i$, where $u_i \in K$ and $w_i \perp K$. Then $v = u + w$, where u is a wedge of $k+1$ vectors in K, and is hence 0, and w is a sum of wedges of terms such as at least one constituent factor per term is perpendicular to K. So $\Lambda^k A v = 0 + \Lambda^k A w = 0$. So $\Lambda^k A \equiv 0$ if $k > n$. Therefore the sum $\sum_{k=0}^{\infty} \mathrm{Tr}(\Lambda^k A)$ only goes up to $k = n$. Suppose $e_1, \ldots, e_n, e_{n+1}, \ldots$ is an orthonormal basis of H such that $e_1, \ldots, e_n$ is an orthonormal basis of K. Reall that

$$\Lambda^n (1 + A) e_1 \wedge \cdots \wedge e_n = \det_{\mathrm{us}}((1 + A)|_K) e_1 \wedge \cdots \wedge e_n.$$

On the other hand

$$\Lambda^n (1 + A) e_1 \wedge \cdots \wedge e_n = (1 + A) e_1 \wedge \cdots \wedge (1 + A) e_n.$$

In the expansion wedge product, each term is a wedge of factors of the form Ae_i or e_j. Set $B^0 = 1$ and $B^1 = A$. Let $\sigma \subseteq \{1, \ldots, n\}$, and interpret $\sigma : \{1, \ldots n\} \to \{0, 1\}$, where $\sigma(i) = 1$ if $i \in \sigma$. Then

$$\Lambda^n (1 + A) e_1 \wedge \cdots \wedge e_n = \sum_{\sigma} B^{\sigma(1)} e_1 \wedge \cdots \wedge B^{\sigma(n)} e_n.$$

For a fixed $\sigma = \{i_1, \cdots, i_k\} \subseteq \{1, \ldots, n\}$, with $\{1, \ldots, n\} \setminus \sigma = \{j_{k+1}, \ldots, j_n\}$, the corresponding term above is equal to

$$\pm A e_{i_1} \wedge \cdots A e_{i_k} \wedge e_{j_{k+1}} \wedge \cdots \wedge e_{j_n} = \pm (\Lambda^k A)(e_{i_1} \wedge \cdots \wedge e_{i_k}) \wedge (e_{j_{k+1}} \wedge \cdots \wedge e_{j_n}), \tag{2.1}$$

with the sign $\pm$ depending on how many swaps are required to turn $e_{\sigma(1)} \wedge \cdots \wedge e_{\sigma(n)}$ into $e_{i_1} \wedge \cdots \wedge e_{i_k} \wedge e_{j_{k+1}} \wedge \cdots \wedge e_{j_n}$. Let us assume without loss of generality that $i_1 < \cdots < i_k$, $j_1 < \cdots < j_k$. Expanding in an orrthonormal basis, we may write

$$(\Lambda^k A)(e_{i_1} \wedge \cdots \wedge e_{i_k}) = \sum_{\ell_1 < \cdots < \ell_k} \langle (\Lambda^k A)(e_{i_1} \wedge \cdots \wedge e_{i_k}), e_{\ell_1} \wedge \cdots \wedge e_{\ell_k} \rangle e_{\ell_1} \wedge \cdots \wedge e_{\ell_k}. \tag{2.2}$$

Let us examine the term corresponding to $\{\ell_1 < \cdots < \ell_k\}$ in this sum. If any $\ell_p > n$, then this term is 0, since A is 0 on the orthocomplement to K. If $\ell_p = j_r$ for some p and r, then the wedge product of this term with $e_{j_{k+1}} \wedge \cdots \wedge e_{j_n}$ is 0. Thus the only term in (2.2) which survives after wedging with $e_{j_{k+1}} \wedge \cdots \wedge e_{j_n}$ is the term corresponding to $\ell_p = i_p$ for all p. Plugging (2.2) into (2.1) and using this fact yields

$$\begin{aligned}
&\pm Ae_{i_1} \wedge \cdots Ae_{i_k} \wedge e_{j_{k+1}} \wedge \cdots \wedge e_{j_n} \\
&= \pm\langle(\Lambda^k A)(e_{i_1} \wedge \cdots \wedge e_{i_k}), e_{i_1} \wedge \cdots \wedge e_{i_k}\rangle e_{i_1} \wedge \cdots \wedge e_{i_k} \wedge e_{j_{k+1}} \wedge \cdots \wedge e_{j_n} \\
&= \langle(\Lambda^k A)(e_{i_1} \wedge \cdots \wedge e_{i_k}), e_{i_1} \wedge \cdots \wedge e_{i_k}\rangle e_1 \wedge \cdots \wedge e_n.
\end{aligned}$$

Since

$$\langle(\Lambda^k A)(e_{\ell_1} \wedge \cdots \wedge e_{\ell_k}), e_{\ell_1} \wedge \cdots \wedge e_{\ell_k}\rangle = 0$$

if any $\ell_p > n$, summing

$$\langle(\Lambda^k A)(e_{i_1} \wedge \cdots \wedge e_{i_k}), e_{i_1} \wedge \cdots \wedge e_{i_k}\rangle e_1 \wedge \cdots \wedge e_n$$

over all subsets $\sigma = \{i_1 < \cdots < i_k\} \subseteq \{1, \ldots, n\}$ is the same as summing it over all sets $\{i_1 < \cdots < i_k\} \subseteq \mathbf{N}$, and thus the sum equals

$$\operatorname{Tr}(\Lambda^k A) e_1 \wedge \cdots \wedge e_n.$$

Recalling the definition of B^j, we have thus shown that

$$\sum_{\#\sigma = k} B^{\sigma(1)} e_1 \wedge \cdots \wedge B^{\sigma(n)} e_n = \operatorname{Tr}(\Lambda^k A) e_1 \wedge \cdots \wedge e_n.$$

The sum of this over all $k \leq n$ is thus on the one had equal to $\det_{\text{us}}((1+A)|_K) e_1 \wedge \cdots \wedge e_n$, as we have shown, and is on the other hand equal to $\left(\sum_{k=0}^n \operatorname{Tr}(\Lambda^k A)\right) e_1 \wedge \cdots \wedge e_n = \det_{\text{Frd}}(1 + A)$. ∎

We will use proposition 2.2 to approximate the Fredholm determinant of an operator by finite-rank approximations. Fortunately, we have lemma 1.5 which will guarantee that the finite-dimensional approximations converge in the trace-class norm. Using the continuity of $\Lambda^k : \ell_1(H) \to \ell_1(\Lambda^k H)$ will allow us to show that the Fredholm determinant is continuous, and thus the finite-dimensional approximations converge. Indeed:

Lemma 2.3. *The Fredholm determinant is continuous in the trace-class norm. Explicitly, if A and B are trace class, then*

$$|\det(1 + A) - \det(1 + B)| \leq \|A - B\|_1 \exp(\max(\|A\|_1, \|B\|_1)).$$

Proof. This follows easily from theorem 1.3. Indeed,

$$|\det(1 + A) - \det(1 + B)| \leq \sum_{k>1} |\operatorname{Tr}(\Lambda^k A - \Lambda^k B)| \leq \sum_{k>1} \|A - B\|_1 \frac{\max(\|A\|_1, \|B\|_1)^{k-1}}{(k-1)!}$$

(the $k = 0$ term vanishes since $\operatorname{Tr}(\Lambda^0 A) = \operatorname{Tr}(\Lambda^0 B) = 1$). The lemma follows. ∎

Theorem 2.4 (Properties of the determinant)**.** *Suppose A, B are trace class. Then*

- *(i)* $\det(1 + A^*) = \overline{\det(1 + A)}$*;*
- *(ii)* $\det(1 + A)\det(1 + B) = \det((1 + A)(1 + B))$*;*
- *(iii) if A is self-adjoint with eigenvalues $\lambda_1, \lambda_2, \ldots$, then* $\det(1 + A) = \prod_{i=1}^\infty (1 + \lambda_i)$*;*
- *(iv) if X is invertible, then* $\det(1 + XAX^{-1}) = \det(1 + A)$*;*
- *(v)* $\det(1 + A) = 0$ *if and only if* $1 + A$ *is not invertible;*
- *(vi)* $\exp(A) - 1$ *is trace class and* $\det(\exp(A)) = \exp(\operatorname{Tr}(A))$.

Proof. Let e_1, e_2 be an orthonomal basis for H and let Π_n be the orthogonal projection onto $\text{span}\{e_1, \ldots, e_n\}$.

Let us first prove (i). For each n, oberserve that

$$((1 + \Pi_n A \Pi_n)|_{\text{range}(\Pi_n)})^* = (1 + \Pi_n A^* \Pi_n)|_{\text{range}(\Pi_n)}.$$

It follows from proposition 2.2 that

$$\begin{aligned} \det(1 + \Pi_n A^* \Pi_n) &= \det_{\text{us}}(((1 + \Pi_n A \Pi_n)|_{\text{range}(\Pi_n)})^*) \\ &= \overline{\det_{\text{us}}((1 + \Pi_n A \Pi_n|_{\text{range}(\Pi_n)})} = \overline{\det(1 + \Pi_n A \Pi_n)}. \end{aligned} \tag{2.3}$$

By lemma 1.5, $\Pi_n A \Pi_n$, and $\Pi_n A^* \Pi_n$ converge to A and A^*, respectively, in the trace class norm, and thus by lemma 2.3, $\det(1+\Pi_n A \Pi_n) \to \det(1+A)$ and similarly $\det(1+\Pi_n B \Pi_n) \to \det(1 + B)$. Taking limits in (2.3) proves (i).

Now let us show (ii). Again from proposition 2.2, for $n \geq N$

$$\det(1 + \Pi_n A \Pi_n) \det(1 + \Pi_n B \Pi_n) = \det(1 + \Pi_n A \Pi_n + \Pi_n B \Pi_n + \Pi_n A \Pi_n B \Pi_n).$$

As above, the left-hand side converges to $\det(1 + A) \det(1 + B)$. For the right-hand side, we know that $\Pi_n A \Pi_n$ and $\Pi_n B \Pi_n B$ converge to A and B in the trace-class norm, so to establish that the right-hand side converges to $\det(1 + A + B + AB) = \det((1 + A)(1 + B))$, we just need to show that $\Pi_n A \Pi_n B \Pi_n \to AB$ in the trace-class norm. Indeed, we may bound

$$\|\Pi_n A \Pi_n B \Pi_n - AB\|_1 \leq \|(\Pi_n - 1)A\|_1 \|\Pi_n B \Pi_n\| + \|A(\Pi_n - 1)\|_1 \|B \Pi_n\| + \|A\| \|B(\Pi_n - 1)\|_1 \to 0.$$

Now let us show (iii). Assume without loss of generality that $e_1, e_2 \cdots$ are eigenvectors of A, and that $Ae_i = \lambda_i e_i$. Then from proposition 2.2

$$\det(1 + \Pi_n A \Pi_n) = \prod_{i=1}^{n} (1 + \lambda_i).$$

Taking $n \to \infty$ as usual (and using that $\sum |\lambda_i| < \infty$) shows (iii).

Next let us show (iv). Let $K_n = \text{range}(\Pi_n)$, and let Γ_n be the orthogonal projection onto $K_n' = X(K_n)$. We know from proposition 2.2 that

$$\begin{aligned} \det(1 + \Pi_n A \Pi_n) &= \det_{\text{us}}((1 + \Pi_n A \Pi_n)|_{K_n}) \\ &= \det_{\text{us}}(X|_{K_n}(1 + \Pi_n A \Pi_n)|_{K_n} X^{-1}|_{K_n'}) \\ &= \det_{\text{us}} t((1 + X \Pi_n A \Pi_n X^{-1})|_{K_n'}) \\ &= \det(1 + (\Gamma_n X \Pi_n A \Pi_n X^{-1} \Gamma_n)|_{K_n'}) \qquad = \det(1 + \Gamma_n X \Pi_n A \Pi_n X^{-1} \Gamma_n). \end{aligned}$$

As usual, the left-hand side converges to $\det(1 + A)$, and the right-hand side converges to $\det(1 + XAX^{-1})$ provided $T_n := \Gamma_n X \Pi_n A \Pi_n X^{-1} \Gamma_n$ converges in the trace-class norm to XAX^{-1}. As in the proof of lemma 1.5, we may take an orthornormal basis $f_1, f_2, \ldots$ such that $f_1, \ldots, f_n$ is a basis of K_n', and thus Γ_n is the orthogonal projection onto $\{f_1, \ldots, f_n\}$. Observe that by definition $\Gamma_n X \Pi_n = X \Pi_n$. Therefore

$$\begin{aligned} \|T_n - XAX^{-1}\|_1 \leq & \|X\| \|(\Pi_n - 1)A\|_1 \|\Pi_n X^{-1} \Gamma_n\| \\ & + \|X\| \|A(\Pi_n - 1)\|_1 \|X^{-1} \Gamma_n\| + \|X\| \|AX^{-1}(1 - \Gamma_n)\|_1 \to 0 \end{aligned}$$

(recall that AX^{-1} is trace class). This shows (iv).

Finally we show (v). Suppose $1 + A$ is not invertible. Since $1 + A$ is Fredholm of index 0, it follows that $1 + A$ has closed range, and $\dim \ker(1 + A) = \dim \text{range}(1 + A)^{\perp}$. In particular, $1+A$ has a null space containing at least one unit-norm vector e_1. Extend e_1 to an orthonormal basis $e_1, e_2, \ldots$ of H. Let Π_n be the projection onto $e_1, \ldots, e_n$. By assumption, $Ae_1 = -e_1$. Thus $\Pi_n A \Pi_n e_1 = -e_1$, and so $(1 + \Pi_n A \Pi_n)e_1 = 0$. Thus $0 = \det(1 + \Pi_n A \Pi_n)$. As usual, this converges to $\det(1 + A)$, which shows that it is 0.

Now suppose $\det(1 + A) = 0$. Then, by (i), $\det(1 + A^*) = 0$, and so by (ii), $\det((1 + A)^*(1+A)) = 0$, and thus $\det(1+(A^*A+A^*+A)) = 0$. Write $(A^*A+A^*+A) = P$. Then P is self-adjoint, P is trace class, and $\det(1 + P) = 0$. Thus, by (iii), $\prod_{i=1}^{\infty}(1 + \lambda_i) = 0$, where λ_i are the eigenvalues of P. If none of the λ_i were -1, then since $\sum |\lambda_i| < \infty$ (since P is traceclass), $\prod_{i=1}^{\infty}(1 + \lambda_i) \neq 0$. Thus, at least one of the $\lambda_i = 0$, and so $1 + P$ has non-trivial kernel, and hence $1 + A$ does, too.

Now let us show (vi). By definition

$$\exp(A) - 1 = \sum_{k=1}^{\infty} \frac{A^k}{k!}.$$

Since $\|A^k\|_1 \leq \|A^{k-1}\|\|A\|_1$, this sum converges absolutely in the trace class norm, and thus converges to a trace-class operator. From proposition 2.2 and properties of the validity of the formula in finite dimensions,

$$\det(\exp(\Pi_n A \Pi_n)) = \exp(\mathrm{Tr}(\Pi_n A \Pi_n)).$$

From lemma 1.5, the right-hand side converges. To show the left-hand side converges, we need to show that $\|\exp(A) - 1 - (\exp(\Pi_n A \Pi_n) - 1)\|_1 \to 0$. By definition, we may control this by

$$\sum_{k=1}^{\infty} \frac{\|(\Pi_n A \Pi_n)^k - A^k\|_1}{k!} = \sum_{k=1}^{\infty} \frac{\|(\Pi_n A)^k \Pi_n - A^k\|_1}{k!}.$$

Let us control the numerator of each term. With the usual trick, one has

$$\begin{aligned} \|(\Pi_n A)^k \Pi_n - A^k\|_1 &\leq \sum_{j=0}^{k-1} \|A\|^j \|(\Pi_n - 1)A\|_1 \|\Pi_n A\|^{k-j-1} + \|A\|^{k-1}\|A(1 - \Pi_n)\|_1 \\ &\leq (k+1)\|A\|^k \max(\|(1 - \Pi_n)A\|_1, \|A(1 - \Pi_n)\|_1). \end{aligned}$$

Therefore

$$\|\exp(A) - \exp(\Pi_n A \Pi_n)\|_1 \leq \max(\|(1 - \Pi_n)A\|_1, \|A(1 - \Pi_n)\|_1) \sum_{k=1}^{\infty} \frac{(k+1)\|A\|^k}{k!}).$$

The sum converges, and the factor out front converges to 0 by lemma 1.5, which proves the claim.

□

Let us end this note by briefly addressing derivatives. Suppose $a < b \in \mathbf{R}$ and $A(t)$, $t \in [a, b]$ is a family of trace-class operators, differentiable at $t = t_0$,[3].

Proposition 2.5 (Jacobi's formula)**.** *If* $1 + A(t_0)$ *is invertible, then* $\det(1 + A(t))$ *is differentiable at* $t = t_0$ *and*

$$\det(1 + A(t))'|_{t=t_0} = \det(1 + A(t_0))\,\mathrm{Tr}((1 + A(t_0))^{-1} A'(t_0)).$$

[3]Here, differentiability means that there exists a trace class $A'(t_0)$ such that $A(t_0+h) - A(t_0) = A'(t_0) + R_h$, where $\|R_h\|_1 \in o(h)$

Proof. Without loss of generality, let us assume that $t_0 = 0$. To start off, let us take the special case $A(t) = tB$, for some trace-class B. Then $A'(0) = B$. By definition,

$$\det(1 + tB) = \sum_{k=0}^{\infty} \operatorname{Tr}(\Lambda^k tB).$$

Testing on k-blades, it is clear that $\Lambda^k tB = t^k \Lambda^k B$. Therefore,

$$|\det(1 + tB) - \det(1 + 0) - \operatorname{Tr}(B)| \leq t^2 \sum_{k=2}^{\infty} t^{k-2} \operatorname{Tr}(\Lambda^k B) \leq t^2 \left(\sum_{k=2} \frac{\|B\|_1^k}{k!} \right),$$

which is certainly in $o(t)$ as $t \to 0$. Now assume $A(t)$ is some aribtrary curve differentiable at 0. Since $A(t)$ is differentiable, we may write $A(t) = A(0) + tA'(0) + R_t$, where $\|R_t\|_1 \in o(t)$. Thus,

$$\begin{aligned}(1 + A(0))^{-1}(1 + A(t)) &= (1 + A(0))^{-1}(1 + A(0) + tA'(0) + R_t) \\ &= 1 + t(1 + A(0))^{-1}A'(0) + (1 + A(0))^{-1}R_t\end{aligned}$$

is of the form $1 + K$, where K is trace-class. In particular

$$\begin{aligned}\det(1 + A(t)) &= \det((1 + A(0))(1 + A(0))^{-1}(1 + A(t))) \\ &= \det(1 + A(0)) \det(1 + t(1 + A(0))^{-1}A'(0) + (1 + A(0))^{-1}R_t).\end{aligned}$$

By lemma 2.3,

$$\begin{aligned}|\det(1 + t(1 + A(0))^{-1}A'(0) + (1 + A(0))^{-1}R_t) &- \det(1 + t(1 + A(0))^{-1}A'(0))| \\ &\leq \|(1 + A(0))^{-1}\| o(t) \exp(C_t),\end{aligned}$$

where

$$\begin{aligned}C_t &= \max(\|t(1 + A(0))^{-1}A'(0) + (1 + A(0))^{-1}R_t\|_1 \|t(1 + A(0))^{-1}A'(0)\|_1) \\ &\leq t\|(1 + A(0))^{-1}\|_1(\|A'(0)\|_1 + o(1))\end{aligned}$$

is uniformly bounded as $t \to 0$. This shows that

$$|\det(1 + t(1 + A(0))^{-1}A'(0) + (1 + A(0))^{-1}R_t) - \det(1 + t(1 + A(0))^{-1}A'(0))| \in o(t),$$

and so

$$\begin{aligned}&\det(1 + t(1 + A(0))^{-1}A'(0) + (1 + A(0))^{-1}R_t) - 1 - \operatorname{Tr}((1 + A(0))^{-1}A'(0)) \\ &= \det(1 + t(1 + A(0))^{-1}A'(0)) - 1 - \operatorname{Tr}((1 + A(0))^{-1}A'(0)) + o(t).\end{aligned}$$

But by the special case, this is just in $o(t)$. Thus, $\det((1+A(0))^{-1}(1+A(t))$ is differentiable with derivative $\operatorname{Tr}((1 + A(0))^{-1}A'(0))$, and so

$$\det(1 + A(t)) = \det((1 + A(0)) \det((1 + A(0))^{-1}(1 + A(t))$$

iss differentiable, too, with derivative

$$\det((1 + A(0)) \operatorname{Tr}((1 + A(t_0))^{-1}A'(0)),$$

as desired. ☐